5.2 Global one-dimensional branches

The following gives a global extension of (Λ, κ) from $(0, \varepsilon)$ to $(0, \infty)$ in the $\mathbb{R}$-analytic case.

Theorem 5.2.1 Suppose (G1)-(G3) hold, $\Lambda' \not\equiv 0$ on $(-\varepsilon, \varepsilon)$ and in $\mathbb{R} \times X$ all bounded closed subsets of $\mathfrak{S}$ are compact. Then there exists a continuous curve $\mathfrak{R}$ which extends $\mathcal{R}^+$ such that

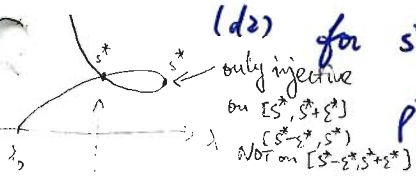
(a) $\mathfrak{R} = \{(\Lambda(s), \kappa(s)) : s \in [0, \infty)\} \subset U$, where $(\Lambda, \kappa) : [0, \infty) \to \mathbb{R} \times X$ is conts.

(b) $\mathcal{R}^+ \subset \mathfrak{R} \subset \mathfrak{S}$.

(c) The set $\{ s \geq 0 : \ker\left(\partial_x F[\Lambda(s), \kappa(s)]\right) \neq \{0\} \}$ has no accumulation points.

(d) At each point, $\mathfrak{R}$ has a local analytic re-parametrization:

(d1) at right nbhd of 0, $\mathfrak{R}$ coincides with $\mathcal{R}^+$;



(d2) for $s^* \in (0, \infty)$, $\exists\ \rho^* : (-1, 1) \to \mathbb{R}$ injective, continuous s.t. $\rho^*(0) = s^*$, $t \mapsto (\Lambda(\rho^*(t)), \kappa(\rho^*(t)))$ is analytic, for $t \in (-1, 1)$.

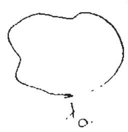
local injective ρ^* moreover, $\exists\ \varepsilon^* > 0$ s.t. $t \mapsto \Lambda(\rho^*(t))$ is injective on $[s^*, s^* + \varepsilon^*]$ and on $[s^* - \varepsilon^*, s^*]$.

(e) One of the following occurs:

(i) $\|(\Lambda(s), \kappa(s))\| \to \infty$ as $s \to \infty$

(ii) $(\Lambda(s), \kappa(s))$ approaches to ∂U as $s \to \infty$

(iii) $\mathfrak{R}$ is a closed loop, i.e. $\exists\ T > 0$ s.t.



$$\mathfrak{R} = \{ (\Lambda(s), \kappa(s)) : 0 \leq s \leq T \}, \text{ with } \Lambda(0) = \lambda_0,\ \kappa(0) = 0.$$

We may assume T is the smallest such number and

$$(\Lambda(s+T), \kappa(s+T)) = (\Lambda(s), \kappa(s)) \quad \forall s \geq 0.$$

H.

(f) If for some $s_1 \neq s_2$,

$(\Lambda(s_1), \kappa(s_1)) = (\Lambda(s_2), \kappa(s_2))$ where $\ker \partial_x F[\Lambda(s_1), \kappa(s_1)] = \{0\}$,

then (e)(iii) occurs and $|s_1 - s_2|$ is an integer multiple of T.

(In particular, $(\Lambda, \kappa) : [0, \infty) \to \mathfrak{S}$ is locally injective.)

Remark 5.2.2

- $\mathfrak{R}$ need not to be a maximal connected subset of $\mathfrak{S}$, others curves or manifolds of $\mathfrak{S}$ may intersect $\mathfrak{R}$.
- $\mathfrak{R}$ is locally injective, but may not be globally injective.
- $\mathfrak{R}$ may have self-intersecting points.
- $\mathfrak{R}$ may not be a smooth curve, but for ε^* sufficiety small, the two segments of $\{(\Lambda(s), \kappa(s)) : 0 < |s - s^*| < \varepsilon^*\}$ (with $(\Lambda(s^*), \kappa(s^*))$ removed) are smooth kurves which can be parametrized by λ.

Bw 5.2.1 The proof will be divided in several steps.

Let $\mathcal{N} := \{ (\lambda, x) \in \mathfrak{S} : \ker \partial_x F[\lambda, x] = \{0\} \}$

— the set of nonsingular solutions of $F(\lambda, x) = 0$ in U.

⌈ for $(\lambda, x) \in \mathcal{N}$, by (G2), $\dim \ker \partial_x F[\lambda, x] = \operatorname{codim} \operatorname{range} \partial_x F[\lambda, x] = 0$

$\Rightarrow \partial_x F[\lambda, x] : X \to Y$ is bijective (and of course continuous)

$\Rightarrow \partial_x F[\lambda, x] : X \to Y$ is a homeomorphism $\xRightarrow{\text{I.F.T.}}$ $\exists_\lambda \in I \subset \mathbb{R}$ and $g : I \to X$ s.t.

locally around (λ, x) in $\mathfrak{S}$ $= \{(\lambda, g(\lambda)) : \lambda \in I\}$

$\Lambda' \not\equiv 0$ on $(-\varepsilon, \varepsilon)$
Λ is $\mathbb{R}$-analytic $\Big\} \Rightarrow \Lambda'(s) \neq 0 \ \forall_{s \in (-\varepsilon,\varepsilon)\setminus\{0\}}$ for ε small.

$$\xrightarrow[4.3.4\,(b)]{\text{Prop}} \mathcal{R}^+ \subset \mathcal{N} \quad \text{for } \varepsilon \text{ small.}$$

Def. 5.2.3 A distinguished arc is a maximal connected subset of $\mathcal{N}$.

for a distinguished arc $\mathcal{L}$ of $\mathcal{N}$, by I.F.T., and (G2),
$\exists$ a (possibly infinite) open interval $I \subset \mathbb{R}$, an $\mathbb{R}$-analytic $g : I \to X$ s.t.

$$\mathcal{L} = \{(\lambda, g(\lambda)) : \lambda \in I\}.$$

Step 1 (L-S reduction) Since structure around points of $\mathcal{N}$ is clear, we consider points of $\mathcal{S} \setminus \mathcal{N}$, to which we want to apply the structure theorem of analytic varieties.

Let $(\lambda_*, x_*) \in \mathcal{S} \setminus \mathcal{N}$. Using Theorem 4.2.1 (L-S red.),
$\exists_{(\lambda_*, 0) \in} V \subset \mathbb{R} \times \ker \partial_x F[\lambda_*, x_*]$, $\psi : V \to X$ $\mathbb{R}$-analytic, and $\mathbb{R}$-analytic $h : V \to \mathbb{R}^q$ for $q = \dim \ker \partial_x F[\lambda_*, x_*]$ such that

(a) $\psi(\lambda_*, 0) = x_*$, $(\lambda, \psi(\lambda, \xi)) \in U$, $\forall (\lambda, \xi) \in V$

(b) $h(\lambda, \xi) = 0 \iff F(\lambda, \psi(\lambda, \xi)) = 0 \quad \forall_{(\lambda,\xi) \in V}$

(c) $\forall (\lambda, x) \in U$ with $F(\lambda, x) = 0$ and $|(\lambda, x) - (\lambda_*, x_*)|$ small, we have
$\exists\, \xi \in \ker \partial_x F[\lambda_*, x_*]$ s.t. $(\lambda, \xi) \in V$ and $x = \psi(\lambda, \xi)$.

(d) $\dim \ker \partial_x F[\lambda, \psi(\lambda, \xi)] = \dim \ker \partial_\xi h[\lambda, \xi] \ \ \forall_{(\lambda,\xi) \in V}$.

Note that $h: V \subset \mathbb{R} \times \ker \partial_x F[\lambda_*, x_*] \simeq \mathbb{R} \times \mathbb{R}^g \to \mathbb{R}^g$ is $\mathbb{R}$-analytic,

by considering its g components : $h = (h_1, h_2, \dots, h_g)$ (each of these components is $\mathbb{R}$-analytic), define

$$A = \operatorname{var}(V, \{h_1, h_2, \dots, h_g\}) = \{(\lambda, \xi) \in V : h(\lambda, \xi) = 0\}$$

— an $\mathbb{R}$-analytic variety in V

$$M = \{(\lambda, \xi) \in V : (\lambda, \psi(\lambda, \xi)) \in \mathcal{N}\}$$

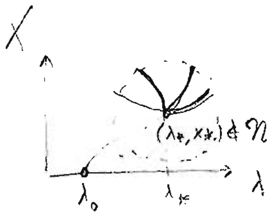
— an $\mathbb{R}$-manifold in V (of dimension 1.)

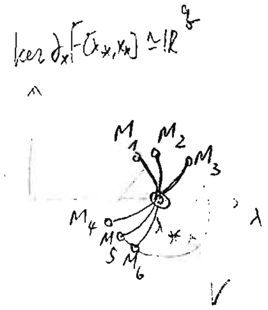
$\ulcorner (\lambda, \psi(\lambda, \xi)) \in \mathcal{N} \Rightarrow 0 = \dim \ker \partial_x F[\lambda, \psi(\lambda, \xi)] = \dim \ker \partial_\xi h[\lambda, \xi] \Rightarrow \partial_\xi h[\lambda, \xi]$ is an isom.

$\Rightarrow$ locally around (λ, ξ) in M, M is a 1-dim. curve passing through (λ, ξ). $\lrcorner$

Thus, M is composed of 1-regular points of A.

Write $\{M_j : j \in J\}$ for the set of non-empty connected components of M.





We assume that $(\lambda_*, 0) \in \overline{M_j}$, $\forall j \in J$ (otherwise, we choose V smaller.)

Complexification of V, h, A, M : (to use the structure theorem)

$h: V \subset \mathbb{R}^{g+1} \to \mathbb{R}^g$

$(x_1, x_2, \dots, x_{g+1}) \mapsto (h_1(x_1, \dots, x_{g+1}), \dots, h_g(x_1, \dots, x_{g+1}))$

where every $h_i = h_i(x_1, \dots, x_{g+1})$ is its Taylor series, a sum of polynomials : $a \cdot x_1^{n_1} x_2^{n_2} \cdot \dots \cdot x_{g+1}^{n_{g+1}}$

$h^c: V^c \subset \mathbb{C}^{g+1} \to \mathbb{C}^g$

$(z_1, z_2, \dots, z_{g+1}) \mapsto (h_1(z_1, \dots, z_{g+1}), \dots, h_g(z_1, \dots, z_{g+1}))$

replacing $\underset{\mathbb{R}}{x}$ with $\underset{\mathbb{C}}{z}$

$a \cdot z_1^{n_1} z_2^{n_2} \cdots z_{g+1}^{n_{g+1}}$

$$A^{c} := \{ (\lambda, \xi) \in V^{c} : h^{c}(\lambda, \xi) = 0 \}$$

$$M^{c} := \{ (\lambda, \xi) \in V^{c} : \ker \partial_{\xi} h^{c}[\lambda, \xi] = \{0\} \}$$

complexifications of A, M.

Write $\{ \hat{M}_i : i \in I \}$ for the set of non-empty connected components of M^{c}.

⌈disconnected sets after complexification may become connected.⌋

Clearly, for every M_j, there is an $i \in I$ such that $M_j \subset \hat{M}_i$.

Step 2 (Application of the structure theorem)

Applying Theorem 5.1.5 on A^{c}, by (a) (b),

$$\gamma_{(\lambda_*, 0)}(A^{c}) = \gamma_{(\lambda_*, 0)}\left(B_1 \cup \dots \cup B_N \cup \{(\lambda_*, 0)\} \right),$$

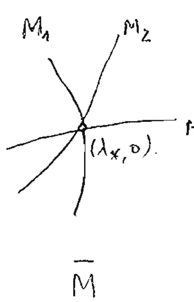
where every B_k is a connected $\mathbb{C}$-analytic manifold of some dimension m_k.

By (d), (e) for every $i \in I$, there is a (real-on-real) branch B_i s.t.

$$\gamma_{(\lambda_*, 0)}(\hat{M}_i) \subset \gamma_{(\lambda_*, 0)}(\overline{B_i}), \quad \text{and} \quad \dim_{\mathbb{C}} B_i = 1, \; B_i \subset A^{c}.$$

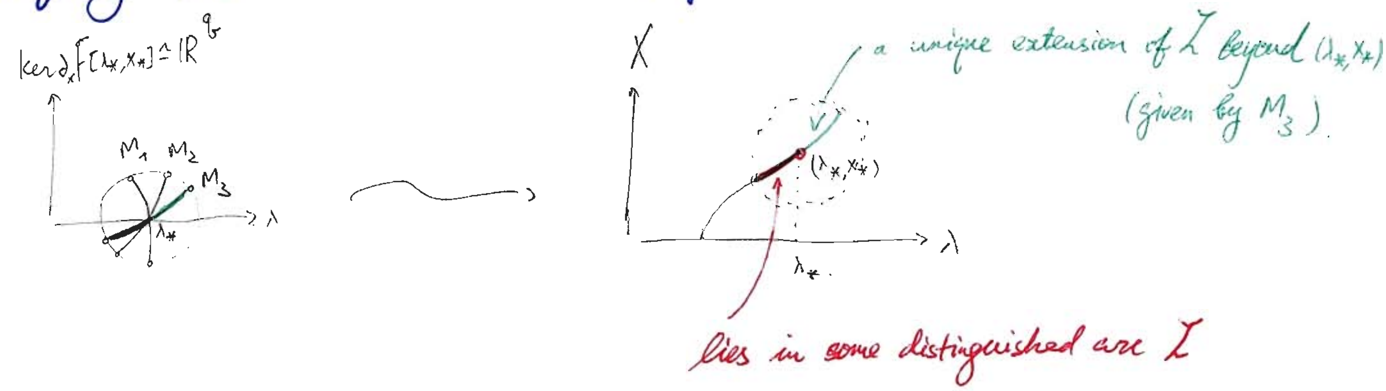
By making V^{c} smaller if necessary, we can assume

$$B_i \setminus \{(\lambda_*, 0)\} \subset \hat{M}_i .$$

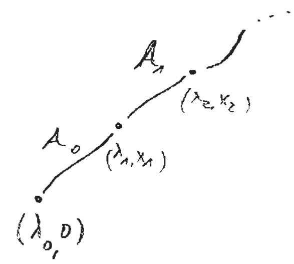


Returning to $\mathbb{R}$-setting, we have $\overline{M}$ is locally around $(\lambda_*, 0)$, a union of finitely many $\mathbb{R}$-analytic curves which pass through $(\lambda_*, 0)$ in V and intersect each other only at $(\lambda_*, 0)$.

Lifting this conclusion to infinite dimensional $\mathbb{R}\times X$:



<u>Def. 5.2.4</u> A <u>route</u> of <u>length</u> $N \in \mathbb{N}\cup\{\infty\}$ is a set $\{\mathcal{A}_n : 0 \le n < N\}$ of distinguished arcs and a set $\{(\lambda_n, x_n) : 0 \le n < N\} \subset \mathbb{R}\times X$ such that



(a) $(\lambda_0, x_0) = (\lambda_0, 0)$ is the bifurcation point;

(b) $\mathcal{R}^+ \subset \mathcal{A}_0$;

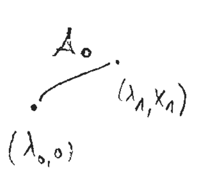
(c) $(\lambda_{n+1}, x_{n+1}) \in (\partial\mathcal{A}_n \cap \partial\mathcal{A}_{n+1}) \setminus \{(\lambda_n, x_n)\}$ for $0 \le n < N-1$, and

$\exists$ injective $\mathbb{R}$-analytic $\rho : (-1,1) \to \mathcal{A}_n \cup \mathcal{A}_{n+1} \cup \{(\lambda_{n+1}, x_{n+1})\}$ with $\rho(0) = (\lambda_{n+1}, x_{n+1})$

⌜So $\mathcal{A}_{n+1}$ is uniquely determined by $\mathcal{A}_n$ by analycity of ρ. ⌟

(d) $n \mapsto \mathcal{A}_n$ is injective.

<u>Step 3</u> (Existence of a maximal route)



I.F.T. $\Rightarrow$ existence of $\mathcal{A}_0$ from $(\lambda_0, 0)$, whose other end point is denoted by (λ_1, x_1)

1-dim (by local theory)

$\mathcal{A}_0 \xrightarrow{\text{uniquely determines}} \mathcal{A}_1 \xrightarrow{\text{uniquely}} \mathcal{A}_2 \; \cdots$ existence ✓

Denote this max. route (extended from $\mathcal{A}_0$) by

$$\{(\mathcal{A}_n, (\lambda_n, x_n)) : 0 \le n < N\}$$

Step 4 (Parametrization of a max. route)

Since every distinguished arc A_n can be parametrized by an $\mathbb{R}$-analytic function, say $(\Lambda_n, \kappa_n) : (n, n+1) \to X$ s.t.

$$A_n = \{(\Lambda_n(s), \kappa_n(s)) : n < s < n+1\}$$

and every "joint" (λ_{n+1}, x_{n+1}) of A_n and A_{n+1}, the parametrization of A_n can be analytically extended to A_{n+1} (because locally around (λ_{n+1}, x_{n+1}), there is an analytic curve structure).

So there is the desired parametrization of

$$\partial\mathcal{R} := \bigcup_{0 \le n < \infty} \left(A_n \cup \{(\lambda_n, x_n)\}\right) \subset \overline{\mathcal{R}}$$

$\boxtimes$ (a)(b)(c)(d)

(e): We distinguish 3 cases:

(1) $N = \infty$
(2) $N < \infty$ and $\overline{A_{N-1}}$ is not compact subset of U $\Big\}$ (i) or (ii) occurs
(3) $N < \infty$ and $\overline{A_{N-1}}$ is compact subset of U. $\}$ (iii) occurs

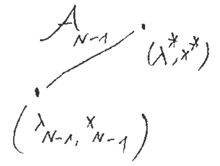
Assume neither (i) nor (ii) happens, then

$\exists \{t_k\}_{k=1}^{\infty}$ with $t_k \to N$ s.t. $\{\Lambda(t_k), \kappa(t_k)\}$ is bounded in $\mathbb{R} \times X$ and bounded away from ∂U.

$\xrightarrow[\text{assumption of 5.2.1}]{\text{compactness}}$ $\{\Lambda(t_k), \kappa(t_k)\}$ is rel. compact, say $(\Lambda(t_k), \kappa(t_k)) \to (\lambda^*, x^*) \in \mathcal{S}$.

If $N=\infty$, then every nbhd of (λ^*, x^*) intersect infinitely many distinct distinguished arcs. $\lightning$ in a nbhd of (λ^*, x^*), the solution set is an $\mathbb{R}$-analytic variety.

If $N<\infty$ and $\overline{A_{N-1}}$ is not compact in U, then (since $t_k \to N$)



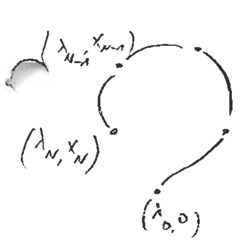
$$(\lambda^*, x^*) \in \partial A_{N-1} \setminus \{(\lambda_{N-1}, x_{N-1})\},$$

thus the route can be extended beyond (λ^*, x^*) $\lightning$ max. route.

$\boxed{\equiv}$ in cases (1), (2), either (i) or (ii) occurs.

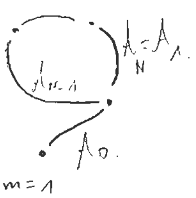
For the case (3), if $N<\infty$ and $\overline{A_{N-1}}$ is compact in U,

let (λ_{N-1}, x_{N-1}) and (λ_N, x_N) be the end points of A_{N-1},



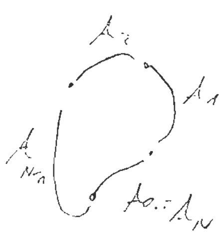
since the route around (λ_N, x_N) can be extended further and the route is already maximal,

$\overset{\text{Def 5.2.4.}}{\underset{(d)}{\Longrightarrow}}$ $A_N = A_m$ for some $m \in \{0, 1, \dots, N-2\}$



We show: $m=0$.

If $m \in \{1, \dots, N-2\}$, then A_{N-1} is extension of $A_N = A_m$ but A_{m-1} (resp. A_{m+1}) is also extension of A_m,



$\Rightarrow A_{N-1} = A_{m-1}$ or A_{m+1} $\lightning$ Def. 5.2.4. (d). $\boxed{\equiv}_{m=0}$.

Thus, $A_0 \dots A_N$ form a closed loop, (e)(iii) occurs $\boxed{\equiv}$(e).

(f): If there are $s_1 \neq s_2$ with

$$(\Lambda(s_1), \kappa(s_1)) = (\Lambda(s_2), \kappa(s_2)) \quad \text{and} \quad \ker \partial_x F[\Lambda(s_1), \kappa(s_1)] = \{0\}$$

then $(\Lambda(s_1), \kappa(s_1)) \in \mathcal{A}_{n_1}$ and $(\Lambda(s_2), \kappa(s_2)) \in \mathcal{A}_{n_2}$ for some $0 \leq n_1, n_2 < N$

$\overset{\text{I.F.T.}}{\Longrightarrow}$ $\mathcal{A}_{n_1}, \mathcal{A}_{n_2}$ coincide in a nbhd of $(\Lambda(s_1), \kappa(s_1)) = (\Lambda(s_2), \kappa(s_2))$

$\Rightarrow$ $\mathcal{A}_{n_1} = \mathcal{A}_{n_2}$ $\Rightarrow$ (e)(iii) occurs and $|s_2 - s_1|$ is a multiple of T.

$\boxed{\Xi}$

5.2.1.